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# Non-lattice determinations of the light quark masses

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The sum rule determinations of $m_s$ and the chiral perturbation theory results for the ratios $m_u/m_d$ and $m_s/m_d$ are reviewed and a method for the extrapolation of lattice data to the physical values of $m_u$ and $m_d$ is outlined.

## 1. INTRODUCTION

The first crude estimates [1] for the magnitude of the three lightest quark masses appeared 25 years ago (values in MeV):

$$m_u \simeq 4 , \quad m_d \simeq 6 , \quad m_s \simeq 135 , \quad [2]$$
$$m_u \simeq 4.2 , \quad m_d \simeq 7.5 , \quad m_s \simeq 150 . \quad [3]$$

Many papers dealing with the pattern of quark masses have been published since then.[1]

One of the reasons for being interested in an accurate determination of the mass values is that these are not understood at all – we need to know the numbers to test ideas that might lead to an understanding of the pattern, such as the relations with the lepton masses that emerge from attempts at unifying the electroweak and strong forces. Another reason is that the Standard Model must describe the low energy properties of the various particles to an amazing degree of accuracy. At low energies, the weak interactions are frozen and the neutrini decouple – the Standard Model reduces to the gauge field theory of SU(3)×U(1). In the framework of this effective theory, the quark masses occur as parameters in the Lagrangian, together with the coupling constants $e, g$, the vacuum angle $\theta$ and the masses of the charged leptons. This Lagrangian is supposed to describe the low energy structure of cold mat-

[1]For an extensive list of refs., see [4]. Like most of these, the present brief report is based on the hypothesis that the quark condensate represents the leading order parameter of the spontaneously broken symmetry. The underlying physics issues are discussed in [5] and the refs. therein.

ter to a very high degree of precision, provided the parameters occurring therein are accurately known. The size of the atoms, for instance, is determined by $a_B = 4\pi/(e^2 m_e)$ and thus only involves parameters for which this is the case.

In the following, I discuss the three light quark masses in terms of the magnitude of $m_s$ and of the ratios $m_u/m_d$ and $m_s/m_d$, which characterize their relative size.

## 2. MAGNITUDE OF $m_s$

Apart from the lattice approach, the best determinations of the magnitude of $m_s$ rely on QCD sum rules [6]. For a recent, comprehensive review, I refer to [7]. A detailed discussion of the method in application to the mass spectrum of the quarks was given nearly 20 years ago [1]. The result for the $\overline{\text{MS}}$ running mass at scale $\mu = 1$ GeV obtained at that time was $m_s(1\,\text{GeV}) = 175 \pm 55$ MeV.

The issue has been investigated in considerable detail since then: Various versions of the sum rules for the pseudoscalar and scalar correlation functions were studied, as well as sum rules for two-point functions formed with baryonic currents. More recently, an entirely independent sum rule determination of $m_s$ based on the strangeness content of the final state in $\tau$-decay became possible – in principle, this source of information is the cleanest one, because the hadronic input does not involve theoretical models, but can be taken from experiment. The analysis of the $\tau$ decay data described in [8] leads

to[2] $m_s(2\,\text{GeV}) = 114 \pm 23\,\text{MeV}$. As discussed there and in [9], this result supersedes the earlier one in [10], which is based on the same experiment and on the same theoretical framework.

The various sum rule results for $m_s$ are reviewed in [7,11]. In [11], the "world average" is quoted as $m_s(2\,\text{GeV}) = 118.9 \pm 12.2\,\text{MeV}$, corroborating the above numbers with a significantly smaller error bar. It is notoriously difficult, however, to account for the systematic errors of the various investigations [12] – in my opinion, the number given does not fully cover these.

The results based on the lattice approach, which were reported at this conference, cluster around $m_s(2\,\text{GeV}) \simeq 90\,\text{MeV}$, that is at the lower edge of the range obtained on the basis of sum rules. I think that the results are consistent with one another. In my opinion, this is important, because it shows that the magnitude of $m_s$ is understood – we should not be content with numerical simulations of the theory, even if these will in the long run yield the most accurate results. A problem would arise, however, if the gradual reduction of the systematic errors to be attached to the lattice results should turn out to push the result downwards – that would be difficult to reconcile with the sum rules, in particular also in view of the lower bounds derived from the positivity of the spectral functions [13].

## 3. PSEUDOSCALAR MASSES

The best determinations of the *relative* size of $m_u$, $m_d$ and $m_s$ rely on the fact that these masses happen to be small, so that the properties of the theory may be analyzed by treating the quark mass term in the Hamiltonian of QCD as a perturbation. The Hamiltonian is split into two pieces:

$$H_{\text{QCD}} = H_0 + H_1 \ ,$$

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where $H_0$ describes the three lightest quarks as massless and $H_1$ is the corresponding mass term,

$$H_1 = \int d^3x \{ m_u \overline{u}u + m_d \overline{d}d + m_s \overline{s}s \} \ .$$

$H_0$ is invariant under the group $\text{SU}(3)_\text{R} \times \text{SU}(3)_\text{L}$ of independent flavour rotations of the right- and lefthanded quark fields. The symmetry is broken spontaneously: The eigenstate of $H_0$ with the lowest eigenvalue, $|0\rangle$, is invariant only under the subgroup $\text{SU}(3)_\text{V} \subset \text{SU}(3)_\text{R} \times \text{SU}(3)_\text{L}$. Accordingly, the spectrum of $H_0$ contains eight Goldstone bosons, $\pi^\pm, \pi^0, K^\pm, K^0, \overline{K}^0, \eta$. The remaining levels form degenerate multiplets of $\text{SU}(3)_\text{V}$ of non-zero mass.

The perturbation $H_1$ splits the SU(3) multiplets, in particular also the Goldstone boson octet. To first order, the squares of the meson masses are linear in $m_u, m_d, m_s$ and the symmetry fixes the coefficients up to a constant:

$$M_{\pi^+}^2 = (m_u + m_d)B_0 + O(m^2) \ ,$$
$$M_{K^+}^2 = (m_u + m_s)B_0 + O(m^2) \ ,$$
$$M_{K^0}^2 = (m_d + m_s)B_0 + O(m^2) \ .$$

In the ratios $M_{\pi^+}^2 : M_{K^+}^2 : M_{K^0}^2$, the constant $B_0$ drops out. Using the Dashen theorem [14] to account for the e.m. self energies (see section 5), these relations imply [3]

$$\frac{m_u}{m_d} \simeq \frac{M_{K^+}^2 - M_{K^0}^2 + M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 0.55 \ ,$$

$$\frac{m_s}{m_d} \simeq \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \simeq 20.1 \ . \qquad (1)$$

## 4. SECOND ORDER MASS FORMULAE

The leading order mass formulae are subject to corrections arising from contributions which are of second or higher order in the perturbation $H_1$. These can systematically be analyzed by means of the effective Lagrangian method [15,16]. In this approach, the quark and gluon fields of QCD are replaced by a set of pseudoscalar fields describing the degrees of freedom of the Goldstone bosons $\pi, K, \eta$. The effective Lagrangian only involves these fields and their derivatives, but contains an infinite string of vertices. For the calculation of

the pseudoscalar masses to a given order in the perturbation $H_1$, however, only a finite subset contributes. The term $\Delta_M$, which describes the SU(3) corrections in the ratio $M_K^2/M_\pi^2$,

$$\frac{M_K^2}{M_\pi^2} = \frac{\hat{m} + m_s}{m_u + m_d}\{1 + \Delta_M + O(m^2)\} \,,$$

involves the two effective coupling constants $L_5$, $L_8$, which occur in the derivative expansion of the effective Lagrangian at first non-leading order [16]:

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{logs} \,.$$

The term $\chi$logs stands for the logarithms characteristic of chiral perturbation theory. They arise because the spectrum of the unperturbed Hamiltonian $H_0$ contains massless particles – the perturbation $H_1$ generates infrared singularities. The coupling constant $L_5$ also determines the SU(3) asymmetry in the decay constants,

$$\frac{F_K}{F_\pi} = 1 + \frac{4(M_K^2 - M_\pi^2)}{F_\pi^2}L_5 + \chi\text{logs} \,.$$

The first order SU(3) correction in the mass ratio $(M_{K^0}^2 - M_{K^+}^2)/(M_K^2 - M_\pi^2)$ turns out to be the same as the one in $M_K^2/M_\pi^2$ [16]:

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\{1 + \Delta_M + O(m^2)\} \,.$$

Hence, the first order corrections drop out in the double ratio

$$Q^2 \equiv \frac{M_K^2}{M_\pi^2} \cdot \frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2} \,. \tag{2}$$

The observed values of the meson masses thus provide a tight constraint on one particular ratio of quark masses:

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}\{1 + O(m^2)\} \,. \tag{3}$$

The constraint may be visualized by plotting the ratio $m_s/m_d$ versus $m_u/m_d$ [17]. Dropping the higher order contributions, the resulting curve takes the form of an ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1 \,, \tag{4}$$

with $Q$ as major semi-axis (the term $\hat{m}^2/m_s^2$ has been discarded, as it is numerically very small).

## 5. VALUE OF $Q$

The meson masses occurring in the double ratio (2) refer to pure QCD. The Dashen theorem states that in the chiral limit, the electromagnetic contributions to $M_{K^+}^2$ and to $M_{\pi^+}^2$ are the same, while the self energies of $K^0$ and $\pi^0$ vanish. Since the contribution to the mass difference between $\pi^0$ and $\pi^+$ from $m_d - m_u$ is negligibly small, the masses in pure QCD are approximately given by

$$(M_{\pi^+}^2)^{\text{QCD}} = (M_{\pi^0}^2)^{\text{QCD}} = M_{\pi^0}^2 \,,$$
$$(M_{K^+}^2)^{\text{QCD}} = M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2 \,,$$
$$(M_{K^0}^2)^{\text{QCD}} = M_{K^0}^2 \,,$$

where $M_{\pi^0}, M_{\pi^+}, M_{K^0}, M_{K^+}$ are the observed masses. Correcting for the electromagnetic self energies in this way, relation (2) yields $Q = 24.2$. For this value of the semi-axis, the ellipse passes through the point specified by Weinberg's mass ratios, eq. (1). The Dashen theorem is subject to corrections from higher order terms in the chiral expansion, which have been analyzed, but are not fully understood. Since the $K^0 - K^+$ mass difference is dominated by the contribution from $m_d - m_u$, the uncertainties in the e.m. part do not very strongly affect the value of $Q$, but they do show up at the 5% level. For a detailed discussion, I refer to [18].

The isospin violating decay $\eta \to 3\pi$ allows one to measure the semi-axis in an entirely independent manner [19]. The transition amplitude is much less sensitive to the uncertainties associated with the electromagnetic interaction than the $K^0 - K^+$ mass difference: The e.m. contribution is suppressed by chiral symmetry and is negligibly small [20]. The transition amplitude thus represents a sensitive probe of the symmetry breaking generated by $m_d - m_u$. In fact, the decay rate is proportional to $Q^{-4}$, with a factor that can be calculated up to corrections of second order in the quark masses. The result for the semi-axis is consistent with the information obtained from $K^0 - K^+$ but is more accurate [21,22]:

$$Q = 22.7 \pm 0.8 \,. \tag{5}$$

## 6. KM–AMBIGUITY

Chiral perturbation theory thus fixes the isospin breaking parameter $m_u/m_d$ in terms of the SU(3) breaking parameter $m_s/m_d$, to within small uncertainties. The ratios themselves, that is the position on the ellipse, are a more subtle issue. Kaplan and Manohar [17] pointed out that the corrections to the lowest order result, eq. (1), cannot be determined on purely phenomenological grounds. They argued that these corrections might be large and that the $u$-quark might actually be massless. A couple of years ago, this possibility was widely discussed in the literature [23], in view of the strong CP problem.

The reason why phenomenology alone does not allow us to determine the two individual ratios beyond leading order is the following. The matrix

$$m' = \alpha_1 m + \alpha_2 (m^+)^{-1} \det m$$

transforms in the same manner as $m$. Symmetry does therefore not distinguish $m'$ from $m$. For a real, diagonal mass matrix, the transformation law for $m_u$, for instance, reads

$$m'_u = \alpha_1 m_u + \alpha_2 \, m_d \, m_s \ . \tag{6}$$

Since the effective theory exclusively exploits the symmetry properties of QCD, the above transformation of the quark mass matrix does not change the form of the effective Lagrangian – the transformation may be absorbed in a suitable change of the effective coupling constants [17]. This implies, however, that the expressions for the masses of the pseudoscalars, for the scattering amplitudes or for the matrix elements of the vector and axial currents, which follow from this Lagrangian, are invariant under the operation $m \to m'$. Conversely, the experimental information on these observables does not distinguish $m_u$ from $m'_u$. Indeed, one readily checks that the transformation $m \to m'$ maps the ellipse onto itself (up to terms of order $(m_u - m_d)^2/m_s^2$, which were neglected). Since the position on the ellipse does not remain invariant, it cannot be extracted from these observables within chiral perturbation theory.

One is not dealing with a hidden symmetry of QCD here – this theory is not invariant under $m \to m'$. Some authors have made the claim that there is no reason for the quark masses entering the Lagrangian of the effective theory to be the same as the running masses of QCD. This claim is incorrect, as can explicitly be demonstrated with the following simple example. The Ward identity for the axial current implies that the vacuum-to-pion matrix element of the pseudoscalar density is given by[3]

$$\langle 0 | \bar{d} \, i \gamma_5 u | \pi^+ \rangle = \sqrt{2} \, F_\pi M_{\pi^+}^2 / (m_u + m_d) \ .$$

The relation is exact, except for electroweak corrections (in the $\overline{\text{MS}}$ scheme, both the matrix element and the quark masses depend on the running scale, but the relation holds at any scale). In the framework of the effective theory, the two sides can separately be calculated in terms of the effective coupling constants and the quark masses occurring in the effective Lagrangian. The relation is obeyed, order by order in the chiral perturbation series. What appears on the right, however, are the quark masses of the effective Lagrangian. The effective theory thus reproduces the matrix element $\langle 0 | \bar{d} \, i \gamma_5 u | \pi^+ \rangle$ if and only if the quark masses in the effective Lagrangian are identified with the running masses of QCD.

## 7. THE RATIOS $m_u{:}\,m_d$ AND $m_s{:}\,m_d$

The KM-ambiguity is of phenomenological nature: Unfortunately, an experimental probe sensitive to the scalar or pseudoscalar currents does not exist – the electromagnetic and weak interactions happen to probe the low energy structure of QCD exclusively through vector and axial currents. This means that theoretical input is needed to determine the size of the two individual mass ratios beyond leading order.

One approach relies on the expansion in $1/N_c$. Since the KM-tranformation (6) violates the Okubo-Iizuka-Zweig rule, the parameter $\alpha_2$ is suppressed in the large $N_c$ limit. In order to cover this limit, the effective theory must be extended, including the degrees of freedom of the $\eta'$ among the dynamical variables, because this particle becomes massless if the number of colours is

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[3]The relation involves the matrix element of a pseudoscalar operator – these are not KM-invariant, even at the level of the effective theory.

sent to infinity and the quark masses are turned off. In fact, within that framework, the KM-ambiguity disappears altogether: The transformation $m' \to m$ preserves the large $N_c$ counting rules only if $\alpha_2$ vanishes to all orders in $1/N_c$ [24]. An evaluation of the mass ratio $m_s/\hat{m}$ based on the large $N_c$ analysis of the transitions $\eta' \to \gamma\gamma$ and $\eta \to \gamma\gamma$ is described in [25].

Another method is based on the hypothesis that the lowest resonances dominate the low energy behaviour of the various Green functions – once the poles and cuts due to the exchange of the Goldstone bosons are accounted for. More precisely, one first shows that the effective coupling constants can be represented in terms of convergent integrals over suitable spectral functions and then assumes that these sum rules are approximately saturated by resonances [26–29].

Both of these methods lead to the conclusion that the corrections to the lowest order mass formulae are small:

$$\frac{m_u}{m_d} = 0.553 \pm 0.043 \; , \quad \frac{m_s}{m_d} = 18.9 \pm 0.8 \; . \qquad (7)$$

For a detailed discussion of this result, I refer to [18]. The values for the ratio $(m_u + m_d)/m_s$ obtained by means of sum rules [11] as well as the lattice results presented at this conference are in remarkably good agreement with these numbers.

## 8. EXTRAPOLATION IN $m_u$ AND $m_d$

The remainder of this report concerns the interface between the lattice approach and the effective theory. As pointed out long ago [30], chiral perturbation theory allows one to predict the dependence of the various correlation functions on the quark masses, as well as on the temperature and on the volume of the box used. In the following, I only discuss the predictions for the dependence on the quark masses.

By now, dynamical quarks with a mass of the order of the physical value of $m_s$ are within reach, but it is notoriously difficult to equip the two lightest quarks with their proper masses. Suppose that we keep $m_s$ fixed at the physical value and set $m_u = m_d = \hat{m}$, but vary the value of $\hat{m}$ in the range $0 < \hat{m} < \frac{1}{2} m_s$ (at the upper end of that range, the pion mass is about 500 MeV).

The expansions of $M_\pi$ and $F_\pi$ in powers of $\hat{m}$ are known to next-to-next-to leading order, from a two loop analysis [31–33] of the effective field theory based on $\mathrm{SU(2)_R} \times \mathrm{SU(2)_L}$:

$$\frac{M_\pi^2}{M^2} = 1 - \tfrac{1}{2}\, x\, \bar{\ell}_3 + \tfrac{17}{8}\, x^2 \bar{\ell}_M^{\,2} + x^2 k_M + O(x^3)\; ,$$

$$\frac{F_\pi}{F} = 1 + x\, \bar{\ell}_4 - \tfrac{5}{4}\, x^2 \bar{\ell}_F^{\,2} + x^2 k_F + O(x^3)\; ,$$

$$\bar{\ell}_M = \frac{1}{51}\left(28\,\bar{\ell}_1 + 32\,\bar{\ell}_2 - 9\,\bar{\ell}_3 + 49\right)\; , \qquad (8)$$

$$\bar{\ell}_F = \frac{1}{30}\left(14\,\bar{\ell}_1 + 16\,\bar{\ell}_2 + 6\,\bar{\ell}_3 - 6\,\bar{\ell}_4 + 23\right)\; ,$$

where $M^2 = 2\,\hat{m}\,B$ is the leading term[4] in the expansion of $M_\pi^2$, while $F$ is the pion decay constant in the limit $\hat{m} = 0$ at fixed $m_s$, and $x$ denotes the dimensionless ratio

$$x = \left(\frac{M}{4\pi F}\right)^2\; . \qquad (9)$$

In this representation, the chiral logarithms are hidden in the quantities $\bar{\ell}_n$, which represent the running coupling constants at scale $\mu = M$ and depend logarithmically on $M$

$$\bar{\ell}_n = \ln \frac{\Lambda_n^2}{M^2}\; . \qquad (10)$$

The mass-independent terms $k_M$ and $k_F$ account for the remainder of $O(M^4)$, in particular for the contributions from the effective couplings of $\mathcal{L}^{(6)}$.

The relations (8) are dominated by the contributions of $O(M^2)$, which involve only two parameters: the values of $\Lambda_3$ and $\Lambda_4$, respectively. In effect, these two parameters replace the coefficients $a_M, a_F$ in the polynomial approximations $M_\pi^2 = M^2(1 + a_M M^2)$, $F_\pi = F(1 + a_F M^2)$ that are sometimes used to perform the extrapolation of lattice data. In contrast to these approximations, the formulae (8) do account for the relevant infrared singularities and are exact, up to and including $O(M^4)$.

## 9. NUMERICAL DISCUSSION

The estimates given in [34] confirm the expectation that the contributions from $k_M$ and $k_F$ are

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[4]Note that $B$ differs from the constant $B_0$ occurring in section 3 by terms of $O(m_s)$.

of order $(M/M_S)^4$, where $M_S \simeq 1\,\mathrm{GeV}$ is the mass scale characteristic of the scalar or pseudoscalar non-Goldstone states contributing to the relevant sum rules (in the SU(2) framework we are using here, the $K\bar{K}$ continuum also contributes to the effective coupling constants, but in view of $4M_K^2 \simeq M_S^2$, the relevant scale is even somewhat larger). Unless the quark masses are taken much larger than in nature, these terms are very small and can just as well be dropped. The relations (8) then specify $M_\pi$ and $F_\pi$ as functions of $\hat{m}$, in terms of the 6 constants $F, B, \Lambda_1, \dots, \Lambda_4$.

Except for $B$ and $\Lambda_3$, all of these can be determined quite accurately from experiment: The scales $\Lambda_1$ and $\Lambda_2$ can be evaluated on the basis of the Roy equations for $\pi\pi$ scattering, while $\Lambda_4$ is related to the scalar charge radius, which can be extracted from a dispersive analysis of the scalar form factor [35,36]. For the purpose of illustration, I use the lowest order relation $B \simeq (M_K^2 - \frac{1}{2}M_\pi^2)/m_s$ to replace the variable $\hat{m}$ by the ratio $\hat{m}/m_s$. For $\Lambda_3$, I invoke the crude estimate $\ln \Lambda_3^2/M_\pi^2 = 2.9\pm2.4$ given in [26], which amounts to $0.2\,\mathrm{GeV} < \Lambda_3 < 2\,\mathrm{GeV}$.

Consider first the ratio $F_\pi/F$, for which the poorly known scale $\Lambda_3$ only enters at next-to-next-to leading order. The upper one of the two shaded regions in fig. 1 shows the behaviour of this ratio as a function of $\hat{m}$, according to formula (8). The change in $F_\pi$ occurring if $\hat{m}$ is increased from the physical value to $\frac{1}{2}m_s$ is of the expected size, comparable to the difference between $F_K$ and $F_\pi$. The curvature makes it evident that a linear extrapolation in $\hat{m}$ is meaningless. The essential parameter here is the scale $\Lambda_4$ that determines the magnitude of the term of order $M^2$. The corrections of order $M^4$ are small – the scale relevant for these is $\Lambda_F \simeq 0.5\,\mathrm{GeV}$.

In the case of the ratio $M_\pi^2/M^2$, on the other hand, the dominating contribution is determined by the scale $\Lambda_3$. The fact that the information about that scale is very meagre shows up through very large uncertainties. In particular, with $\Lambda_3 \simeq 0.5\,\mathrm{GeV}$, the ratio $M_\pi^2/M^2$ would remain very close to 1, on the entire interval shown. The corrections of $O(M^4)$ are small also in this case (the relevant scale is $\Lambda_M \simeq 0.6\,\mathrm{GeV}$).

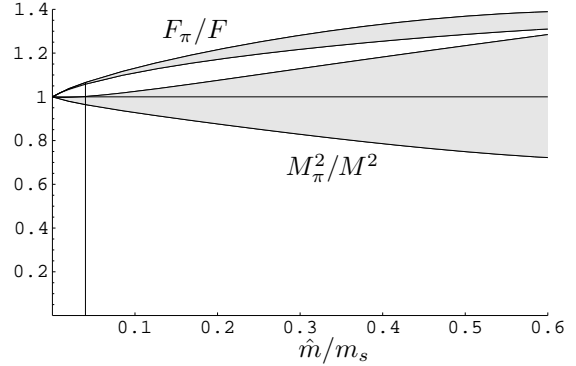The above discussion shows that brute force is



Figure 1. Dependence of the ratios $F_\pi/F$ and $M_\pi^2/M^2$ on $\hat{m} = \frac{1}{2}(m_u + m_d)$. The strange quark mass is held fixed at the physical value. The vertical line corresponds to the physical value of $\hat{m}$.

not the only way to reach the very small values of $m_u$ and $m_d$ observed in nature. It suffices to equip the strange quark with the physical value of $m_s$ and to measure the dependence of the pion mass on $m_u, m_d$ in the region where $M_\pi$ takes a value like 400 or 500 MeV. Since the dependence on the quark masses is known rather accurately in terms of the two constants $B$ and $\Lambda_3$, a fit to the data based on eq. (8) should provide an extrapolation to the physical quark masses that is under good control. Moreover, the resulting value for $\Lambda_3$ would be of considerable interest, because that scale also shows up in other contexts, in the $\pi\pi$ scattering lengths, for example. A measurement of the mass dependence of $F_\pi$ in the same region would be useful too, because it would provide a check on the dispersive analysis of the scalar radius that underlies the determiation of $\Lambda_4$ (in view of the strong unitarity cut in the scalar form factor, a direct evaluation of the scalar radius on the lattice is likely more difficult).

## 10. $\pi\pi$ SCATTERING LENGTHS

As a further example, I consider the two $\pi\pi$ $S$-wave scattering lengths, for which the dependence on the quark masses was discussed already in [37], including a comparison with the quenched lattice data available at the time. The two loop representation [32] explicitly specifies the scattering lengths in terms of the effective coupling constants, up to and including contributions of
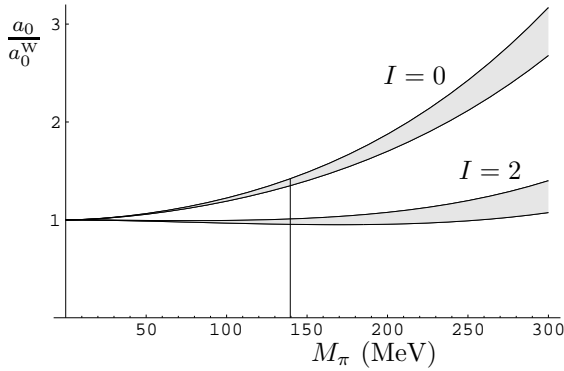
Figure 2. $\pi\pi$ $S$-wave scattering lengths as a function of the pion mass.

$O(M^6)$. Fig. 2 shows the corresponding corrections to Weinberg's low energy theorem[5] [38]

$$I=0: a_0^W = \frac{7M_\pi^2}{32\,\pi\,F_\pi^2}\,, \qquad I=2: a_0^W = -\frac{M_\pi^2}{16\,\pi\,F_\pi^2}\,.$$

Note that the Weinberg formulae are written in terms of the values of $F_\pi$ and $M_\pi$ that correspond to the quark mass of interest – instead of the corresponding lowest order terms $F$ and $M$. Although the range shown here is considerably smaller than in fig. 1, the corrections are much larger: The expansion of the scattering lengths in powers of $M_\pi$ converges only very slowly. The effect arises from the chiral logarithms associated with the unitarity cut, which in the case of the $I=0$ channel happen to pick up large coefficients. In fact, the chiral perturbation theory formulae underlying the figure are meaningful only in the range where the corrections are small (the shaded regions exclusively account for the uncertainties in the values of the coupling constants).

I conclude that, in the case of the $I=0$ scattering length, the extrapolation in $\hat{m}$ requires significantly smaller quark masses than the one for $M_\pi$ or $F_\pi$. In the $I=2$ channel, the effects are much smaller, because this channel is exotic: The final state interaction is weak and repulsive.

The method proposed in [36] replaces the expansion of the scattering amplitude at threshold

by one in the unphysical region, where the convergence properties are similar to those for $F_\pi$ and $M_\pi^2$. Indeed, that method yields a remarkably precise prediction for the scattering lengths: $a_0^0 = 0.220 \pm 0.005$, $a_0^2 = -0.0444 \pm 0.0010$. The lattice result, $a_0^2 = -0.0374 \pm 0.0049$ [39], is on the low side, but not inconsistent with the prediction.

## 11. CONCLUSIONS

The results for $m_s$ obtained from recent lattice simulations with dynamical quarks are consistent with those based on sum rules, but the central values are on the low side.

The lattice results for the ratios $m_u/m_d$ and $m_s/m_d$ confirm the values obtained in the framework of chiral perturbation theory.

The dependence of the various observables on the quark masses contains infrared singularities which can be worked out by means of chiral perturbation theory. This information might be useful for the extrapolation of lattice results to the small quark mass values of physical interest.

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